Technische Universität Ilmenau
Institut für Mathematik

Preprint No. M 04/18

Adaptive sampled-data tracking for input constrained exothermic chemical reaction models

Ilchmann, Achim; Townley, Stuart;
Thuto, Mosalagae

September 2004

Impressum:
Hrsg.: Leiter des Instituts für Mathematik
Weimarer Straße 25
98693 Ilmenau
Tel.: +49 3677 69 3621
Fax: +49 3677 69 3270
http://www.tu-ilmenau.de/ifm/

ISSN xxxx-xxxx

ilmedia

Adaptive sampled-data tracking for input constrained exothermic chemical reaction models

Achim Ilchmann

Institut für Mathematik, Technische Universität Ilmenau, Weimarer Straße 25, 98693 Ilmenau, DE

Stuart Townley

School of Engineering, Computer Science and Mathematics, University of Exeter, North Park Road, Exeter EX4 4QE, UK

Mosalagae Thuto

Department of Mathematics, University of Botswana, Private Bag 0022, Botswana

Abstract

We consider digital input-constrained adaptive output feedback control of a class of nonlinear systems which arise as models for controlled exothermic chemical reactors. Our objective is set-point control of the temperature of the reaction, with pre-specified asymptotic tracking accuracy set by the designer. Our approach is based on λ–tracking controllers, but we introduce a piecewise constant sampled-data output feedback strategy with adapted sampling period. The approach does not require any knowledge of the systems parameters, does not invoke an internal model, is simple in its design, copes with noise corrupted output measurements, and requires only a feasibility assumption in terms of the reference temperature and the input constraints.

Key words: Adaptive control, exothermic chemical reactors, input saturation, sampling, tracking, global stabilisation

Email addresses: ilchmann@mathematik.tu-ilmenau.de (Achim Ilchmann), townley@maths.ex.ac.uk (Stuart Townley), thutomv@mopipi.ub.bw (Mosalagae Thuto).

1 Introduction

We consider adaptive sampled-data control of input constrained continuous-stirred, exothermic chemical reactor models. The main issues of relevance include: typically reactor models are highly nonlinear; the rate of conversion of reactant into product should be economically profitable; quite often the operating point is close to an open-loop unstable hyperbolic equilibrium; there is model uncertainty, inputs are saturated; and measured data can be corrupted by noise. A brief review of the relevant literature can be found in Jadot et al. [6] and Ilchmann et al. [5].
In the latter, we developed simple adaptive controllers for nonlinear reactor models and proved that the controllers could handle the issues mentioned above. However, two open questions remained: (i) Is it possible to prespecify the closed-loop performance? (ii) Can the controller be implemented digitally? The performance issue has been addressed in Ilchmann and Trenn [4] in a context of funnel control; we consider the issue of digital implementation in the current note.

1.1 System class

As in [6] and [5], we consider coupled reactant-product-temperature models of the form

$$\left.\begin{aligned} \dot{x}_1(t) &= C^1\, r\Big(x(t),T(t)\Big) + d\,[v(t) - x_1(t)], && x_1(0) = x_1^0 \in \mathbb{R}^{n-m}_{\geq 0} \\ \dot{x}_2(t) &= C^2\, r\Big(x(t),T(t)\Big) + d\,[x_2^{\text{in}} - x_2(t)], && x_2(0) = x_2^0 \in \mathbb{R}^{m}_{\geq 0} \\ \dot{T}(t) &= b^T\, r\Big(x(t),T(t)\Big) - q\,T(t) + u(t), && T(0) = T^0 \in \mathbb{R}_{>0}\,. \end{aligned}\right\} \quad (1)$$

In (1) the variables and constants represent, for $n, m \in \mathbb{N}$ with $0 < m < n$, the following

$x(t) = \binom{x_1}{x_2} \in \mathbb{R}^n_{\geq 0}$	concentration of chemical reactants and products at time $t \geq 0$,
$x_1(t) \in \mathbb{R}^{n-m}_{\geq 0}$	concentrations of the $n-m$ chemical reactants at time $t \geq 0$,
$x_2(t) \in \mathbb{R}^{m}_{\geq 0}$	concentrations of the m chemical products at time $t \geq 0$,

$T(t) \in \mathbb{R}_{>0}$	temperature of the reactor at time $t \geq 0$,
$x^{\text{in}} = \binom{x_1^{\text{in}}}{x_2^{\text{in}}} \in \mathbb{R}_{\geq 0}^{(n-m)+m}$	constant feed concentrations,
$v(t) \in [0, x_1^{\text{in}}]$	control of the cooling at time $t \geq 0$,
$u(t) \in \mathbb{R}_{\geq 0}$	control of the temperature at time $t \geq 0$,
$C = \binom{C^1}{C^2} \in \mathbb{R}^{n \times m}$	stoichiometric matrix,
$C^1 \in \mathbb{R}_{\leq 0}^{(n-m)\times m}$	stoichiometric matrix of the reactants, accordingly all entries of C^1 are non-positive,
$b \in \mathbb{R}_{\geq 0}^m$	coefficients of the exothermicity,
$d > 0$	dilution rate,
$q > 0$	heat transfer rate between heat exchanger and reactor,
$r(\cdot,\cdot) : \mathbb{R}_{\geq 0}^n \times \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}^m$	a locally Lipschitz continuous function modelling the reaction kinetics.

The system (1) and the following assumptions (A1)–(A4) capture the essential features of exothermic chemical reactor models.

(A1) $\mathbb{R}_{\geq 0}^n \times \mathbb{R}_{>0}$ is positively invariant under (1) for any bounded, nonnegative, piecewise constant functions $u(\cdot) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and $v(\cdot) : \mathbb{R} \to [0, x_1^{\text{in}}]$.

(A2) There exists $\gamma \in \mathbb{R}_{>0}^n$ such that $\gamma^T c_i \ \leq \ 0$ for all columns $c_1, \ldots, c_m$

of the stoichiometric matrix C.

(A3) For $T^* > 0$ there exist $0 < \underline{T} < T^* < \overline{T}$, $\rho > 0$, $0 < \underline{u} < \overline{u}$, such that

$$0 < \underline{u} + \rho < qT - b^T r(x,T) < \overline{u} - \rho \quad \forall\ (x,T) \in \Omega(\gamma, x^{\text{in}}) \times [\underline{T}, \overline{T}].$$

where

$$\boldsymbol{\Omega}(\gamma, \mathbf{x}^{\text{in}}) \ := \ \left\{ x \in \mathbb{R}_{>0}^n \middle|\ \gamma^T x < \gamma^T x^{\text{in}} \right\}$$

(A4) $\|r(x,T)\| \leq \hat{r}(x_1)\, T \quad \forall\ (x,T) = ((x_1^T, x_2^T)^T, T) \in \Omega(\gamma, x^{\text{in}}) \times \mathbb{R}_{>0}$

for some continuous function $\hat{r} : \mathbb{R}_{\geq 0}^{n-m} \to \mathbb{R}_{\geq 0}$ with $\lim\limits_{x_1 \to 0} \hat{r}(x_1) = 0$.

Assumption (A1) is natural for exothermic reactions. Indeed, concentrations and temperature should not become zero once they are positive. In fact, since $r(\cdot,\cdot)$ is nonnegative, if $u(\cdot)$ is nonnegative, then it is clear that $T(t) > 0$ whenever $T^0 > 0$. It is easy to show that the remainder of (A1) holds automatically

when $n = 2$, i.e., in the case of a single reaction. For multiple reactions, there are various conditions (see, e.g., [Proposition 6], Ilchmann and Weirig [3]) in terms of specific rates which imply that (A1) holds.
The assumption (A1) has been formulated for the closed positive orthant $\mathbb{R}^n_{\geq 0}$ of the concentrations and the open half line for the temperature. The latter is natural since the reactor should not operate with zero or negative temperature; the former could also be assumed for the open positive orthant $\mathbb{R}^n_{>0}$; the analysis goes through without any changes.

Assumption (A2) holds if (1) satisfies the law of conservation of mass, which means that there exists $\gamma \in \mathbb{R}^n_{>0}$ with $\gamma^T C = 0$. This can be found implicitly in Gavalas [2], and it is also assumed in Viel et al. [8]. If C does not represent exactly the stoichiometric relationships between all species, then conservation of mass need not be satisfied. Nevertheless, the reaction model might still be relevant provided that all essential reactions are obeyed. This approach was adopted in Bastin and Dochain [1] and also in [3]. A concept of "noncyclic processes" is developed in the latter article to ensure dissipativity of mass, and hence that (A2) is satisfied.

Assumption (A3) guarantees "feasibility" by relating the temperature setpoint T^* and the positive input saturations $\underline{u}$ and $\overline{u}$ to weak system data. This assumption could be formulated less technically as

$$\text{(A3}'\text{)} \quad \begin{array}{l} \text{For } \; T^* > 0 \text{ there exist } 0 < \underline{u} < \overline{u}, \text{ such that} \\ \underline{u} < qT^* - b^T r(x, T^*) < \overline{u} \qquad \forall\, x \in \Omega(\gamma, x^{\text{in}}) \,, \end{array}$$

which implies (A3) for suitable $\underline{T}, \overline{T}, \rho$. However, the more explicit assumption (A3) is easier to use, and the introduction of ρ makes the exposition in the proofs clearer. Note that (A3$'$) coincides with (H3) in [8].

Finally, assumption (A4) encompasses multireaction kinetics considered in [8] and guarantees in particular that the reaction kinetics are zero if the temperature is zero.

1.2 Control objective

The control objective is to regulate the temperature $T(t)$ towards a prespecified neighbourhood of size $\lambda > 0$ of a given reference temperature T^*, whilst maintaining boundedness of all variables. In achieving this objective, we are restricted to using only temperature measurements which may be corrupted by bounded noise $n(\cdot) : \mathbb{R}_{\geq 0} \to \mathbb{R}$, so that the measured error becomes

$$e(t) \;=\; T^* - T(t) + n(t). \tag{2}$$

We stress that the noise need not be smooth but its norm has to be sufficiently small in terms of λ, i.e. the accuracy of the tracking error, and in terms of

$\overline{T} - T^*$. This will be specified in (8) and it is needed basically to ensure that a feedback based on disturbed outputs can recognise the difference between T^* and $\overline{T}$.

Not only do we want to cope with noise corrupted measurements but additionally we want the controller to be implemented digitally so that we have access to these corrupted measurements only at sampled time instants. Specifically, we use zero-order sampling

$$T_i := T(t_i)$$

and zero-order hold

$$u(t) := u(t_i) \quad \text{for all } t \in [t_i, t_{i+1}),$$

where the sampled time is t_i, $i \in \mathbb{N}$, and $t_0 = 0$.

1.3 Adaptive sampled-data controller

For initial values $t_0 = 0$, $\beta_0 > 0$, and input offset $u^* \in (\underline{u}, \overline{u})$, we define the adaptive sample and hold controller, for all $i \in \mathbb{N}_0$, as follows

Sampling

$$e_i \;:=\; e(t_i); \tag{3}$$

Sampling rate adaptation

$$h_i = f(\beta_i)\,, \quad t_{i+1} = t_i + h_i, \tag{4}$$

where $f(\cdot) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ is a non-increasing function with $\lim_{\beta\to\infty} f(\beta) = 0$.

Gain adaptation

$$\beta_{i+1} \;=\; \beta_i + h_i\, g_\lambda(e_i) \tag{5}$$

where, for $\lambda > 0$, the continuous function $g_\lambda : \mathbb{R} \to \mathbb{R}_{\geq 0}$ satisfies

$$\forall\, e \in \mathbb{R} : \; \Big[\; g_\lambda(e) = 0 \iff |e| \leq \lambda \;\Big] \tag{5a}$$

$$\liminf_{e\to\infty} g_\lambda(e) > 0\,; \tag{5b}$$

Zero-order hold feedback

$$\left.\begin{aligned} u(t) &:= u_i = \mathrm{sat}_{[\underline{u},\overline{u}]}\left(\beta_i e_i + u^*\right), \\ v(t) &:= v_i = \begin{cases} 0, & \text{if } \beta_i e_i \leq \underline{u} - u^* \\ x_1^{\mathrm{in}}, & \text{else} \end{cases} \end{aligned}\right\} \quad t \in [t_i, t_{i+1}). \tag{6}$$

An *example* of the control strategy (3)–(6) is, for $p \geq 1$, $\alpha, \beta_0 > 0$, and $t_0 = 0$,

$$\left.\begin{aligned} h_i &= (1+\beta_i)^{-\alpha}, \quad t_{i+1} = t_i + h_i, && e_i = T^* - T(t_i) + n(t_i) \\ \beta_{i+1} &= \beta_i + h_i \max\{|e_i| - \lambda, 0\}^p, && \\ u(t) &= \mathrm{sat}_{[\underline{u},\overline{u}]}\left(\beta_i e_i + u^*\right), && t \in [t_i, t_{i+1}) \\ v(t) &= \begin{cases} 0, & \text{if } \beta_i e_i \leq \underline{u} - u^* \\ x_1^{\mathrm{in}}, & \text{else} \end{cases} && t \in [t_i, t_{i+1}) \end{aligned}\right\} \tag{7}$$

Note the simplicity of the controller's design. It consists merely of a monotonically non-decreasing gain adaptation (5), depending on the distance of the error to the λ-strip; a decreasing sampling adaptation (4), tuned by the gain which increases as long as the measured error e_i is outside the λ-strip; and a zero-order hold feedback (6) which is piecewise constant on the sampling intervals and obeys input saturations. The flexibility of selecting suitable functions f and g_λ could be useful in applications.

2 Main results

We are now in a position to state the main result which, loosely speaking, shows that, under the standard chemical reaction model assumptions (A1)–(A4), temperature setpoint tracking can be achieved with prespecified accuracy $\lambda > 0$ by the simple sampled-data control strategy (3)–(6): the sampled temperature T_i tends to the $[\lambda+\|n\|_\infty]$-strip $\left[T^*-[\lambda+\|n\|_\infty],\, T^*+[\lambda+\|n\|_\infty]\right]$ as $i \to \infty$; moreover, all signal are bounded, and convergence of the gain and sampling period is ensured. We stress that the controller tolerates (not necessarily smooth) noise corrupting the output measurement as long as the noise is sufficiently small.

Theorem 1 (Global result)
Consider (1) satisfying (A1)–(A4), let $\lambda > 0$, and $n(\cdot) : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be a function satisfying

$$2\,\|n\|_\infty \;<\; \min\left\{\overline{T} - T^*,\, \lambda\right\}. \tag{8}$$

Then the application of the sampled-data adaptive controller (3)–(6) to (1) yields, for any initial data $(x^0, T^0, \beta_0) \in \Omega(\gamma, x^{\mathrm{in}}) \times \mathbb{R}_{>0} \times \mathbb{R}_{>0}$, an initial value problem which has a unique solution

$$(x(\cdot), T(\cdot)) : \mathbb{R}_{\geq 0} \longrightarrow \Omega(\gamma, x^{\mathrm{in}}) \times \mathbb{R}_{>0}$$

and this solution satisfies

(i) $T(\cdot)$ is bounded;

(ii) $\beta_i \uparrow \beta_\infty \in \mathbb{R}_{>0}$ as $i \to \infty$;

(iii) $h_i \downarrow h_\infty \in \mathbb{R}_{>0}$ as $i \to \infty$;

(iv) $\lim_{i\to\infty} \operatorname{dist}\Big(|T^* - T_i|,\ [0, \lambda + \|n\|_\infty]\Big) = 0\,.$

Note that if $n(\cdot) \equiv 0$, then (8) is redundant. In this sense, our controller is robust with respect to sufficiently small output measurement noise. Theorem 1 is proved in the Appendix.

In the control strategy (6) we have two control actions: a saturating control of the temperature and an on/off control of reactant feed. If the upper feasibility bound $\overline{T}$ is known, and additionally we have that the initial temperature is below this bound, then we only need to use saturated control $u(\cdot)$ of temperature. In the absence of additional cooling action we then have a model

$$\left.\begin{aligned} \dot{x}(t) &= C\, r\Big(x(t), T(t)\Big) + d\,[x^{\mathrm{in}} - x(t)], && x(0) = x^0 \in \mathbb{R}^n_{\geq 0} \\ \dot{T}(t) &= b^T r\Big(x(t), T(t)\Big) - q\,T(t) + u(t), && T(0) = T^0 \in \mathbb{R}_{>0}\,. \end{aligned}\right\} \tag{9}$$

Proposition 2 (Local result)
Consider (9) satisfying (A1)–(A3), let $\lambda > 0$, and $n(\cdot) : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be a function satisfying (8). Then the application of the sampled-data adaptive controller (3)–(6) to (9) yields, for any initial data $(x_0, T_0) \in \Omega(\gamma, x^{\mathrm{in}}) \times (0, \overline{T}]$, and β_0 sufficiently large, an initial value problem which has a unique solution

$$(x(\cdot), T(\cdot)) : \mathbb{R}_{\geq 0} \longrightarrow \Omega(\gamma, x^{\mathrm{in}}) \times (0, \overline{T})$$

and this solution satisfies the Statements (i)–(iv) of Theorem 1.

For brevity we omit the proof of Proposition 2 which is a simplification of the proof of Theorem 1.

3 Example and simulations

In this section, we consider the special case of a single reaction so that (1) is of the form

$$\left.\begin{aligned} \dot{x}_1(t) &= -k(T(t))\, x_1(t) + d\,[v(t) - x_1(t)], && x_1(0) = x_1^0 \in \mathbb{R}_{\geq 0} \\ \dot{x}_2(t) &= k(T(t))\, x_1(t) + d\,[x_2^{\mathrm{in}} - x_2(t)], && x_2(0) = x_2^0 \in \mathbb{R}_{\geq 0} \\ \dot{T}(t) &= b\, k(T(t))\, x_1(t) - q\,T(t) + u(t), && T(0) = T^0 \in \mathbb{R}_{>0}\,. \end{aligned}\right\} \tag{10}$$

Here $b > 0$ denotes the exothermicity of a reaction $A \longrightarrow B$, $x^{\text{in}} = (x_1^{\text{in}}, 0)^T$, where x_1^{in} is the constant feed rate of reactant A, and the reaction kinetics are given by a locally Lipschitz function $k(\cdot) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $k(0) = 0$. A typical example of $k(\cdot)$ is the Arrhenius law $T \mapsto k_0 \exp\{-E/RT\}$ (extended to zero by continuity), where $k_0 \in \mathbb{R}_{>0}$ is a constant, E is the activation energy, and R is the Joule constant. The parameters of the function $k(\cdot)$ and the positive constants d, q, and b are typically unknown.

This prototypically example is also studied in [8] and in [5]. Here we choose the same system parameters – consistent with a laboratory-scale reaction vessel of approximately 100 liters – so that simulations can be compared.

$$\begin{aligned} &T \mapsto k(T) = k_0\, e^{-k_1 T},\ k_0 = e^{25},\ k_1 = 8700\,[\mathrm{K}], \\ &\ d = 1.1,\ q = 1.25\,[\mathrm{min}^{-1}],\ x_1^{\text{in}} = x_2^{\text{in}} = 1,\ [\mathrm{mol/l}],\ b = 209.2\,[\mathrm{Kl/mol}]. \end{aligned} \tag{11}$$

The controller (7) with parameters

$$\begin{aligned} &\underline{u} = 295,\ \overline{u} = 505,\ u^* = 330,\ T^* = 337.1\,[\mathrm{K}], \\ &\qquad\qquad\qquad\qquad \beta^0 = 12,\ p = 2,\ \alpha = 1,\ \lambda = 2.85\,, \end{aligned} \tag{12}$$

is applied to (10) with initial conditions

$$x_1^0 = 0.02, \quad x_2^0 = 1.07, \quad T^0 \in \{270, 320, 390\}\,, \tag{13}$$

and disturbance signal

$$n(t) = q_1(t)/15, \tag{14}$$

where $q_1(\cdot)$ is the first component of the Lorenz equation

$$\tfrac{\mathrm{d}}{\mathrm{d}t}(q_1, q_2, q_3) = \Big(10[q_2 - q_1],\ 28q_1 - q_2 - q_1 q_3,\ q_1 q_2 - \tfrac{8}{3} q_3\Big), \quad (q_1, q_2, q_3)(0) = (1, 0, 3).$$

This Lorenz equation is known Sparrow [7] to exhibit chaotic but bounded behaviour. In this case $|n(t)| \leq 1.25$ for all $t \geq 0$.

The objective is to regulate the temperature to a neighborhood of $T^* = 337.1[\mathrm{K}]$. It is easy to see that for

$$\gamma = (1, 1)^T, \quad \underline{T} = 240, \quad \overline{T} = 339.65\ [\mathrm{K}], \quad \rho = 5 \tag{15}$$

the inequality in (8) and the feasibility assumption (A3) are satisfied for some $\rho > 0$.

All the above parameters are the same as for the simulations in [5], and the sampled data controller is the Euler approximation of the continuous time controller. The Figures 1 and 2 depict sampled-data simulations corresponding to their continuous time counterparts, which can be seen in Figures 2 and 3 from [5].

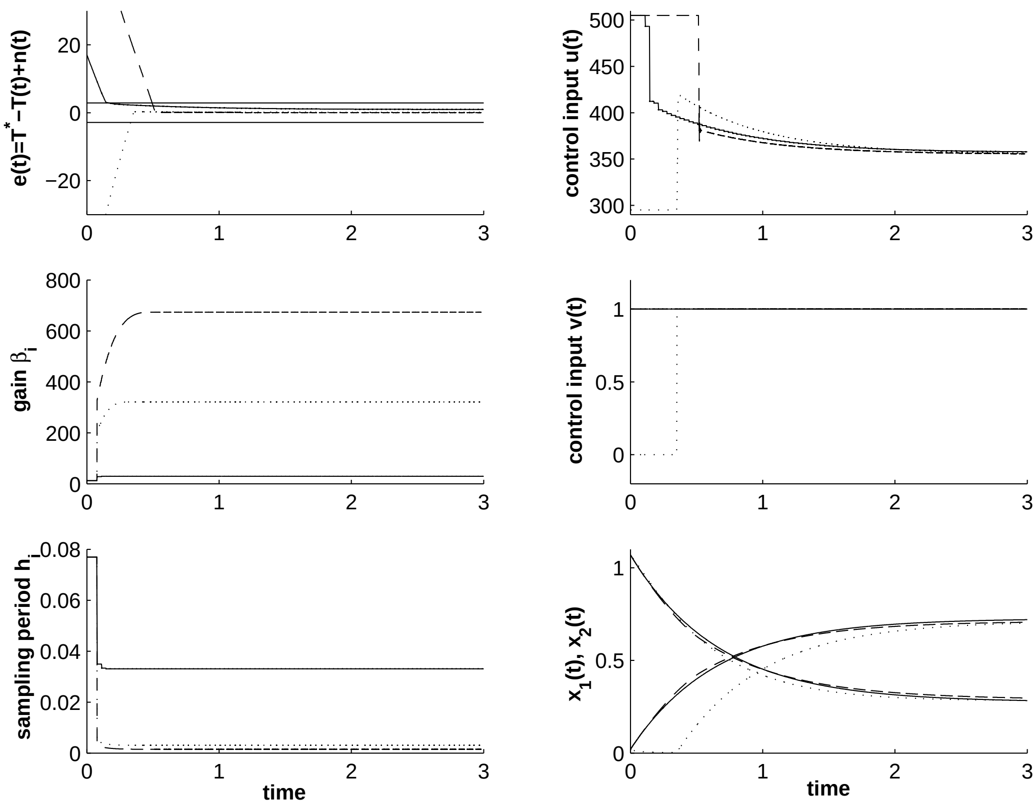

Fig. 1. Closed-loop setpoint control of the adaptive λ-tracker (7) with control parameters (12) applied to the single reaction (10) with parameters (11) and initial temperature $T^0 = 270$ (dashed), $T^0 = 320$ (solid), $T^0 = 390$ (dotted).

In [5] we have shown by simulations that for initial temperature $T^0 = 390 \not\in [0, \overline{T}]$, the local controller cannot cope: there is a thermal runaway and the temperature is attracted to a stable but undesirably high temperature. As a result, the reaction becomes overheated, the reactant burns out, and there is a rapid growth of the product. Furthermore, the control input saturates at its lower limit throughout the simulation and the gain increases unboundedly. The same would be true for the sampled-data result as in Proposition 2. We have omitted this simulation for brevity.

Figure 1 shows the simulations in the noise free case. The final gain is only slightly larger than for continuous time adaptation and can be reduced by exploiting the flexibility of selecting f and g_λ in the admissible gain adaptation. Note that the gain takes on largest values for initial temperature $T^0 = 390$ which is the temperature that cannot be handled by the corresponding local adaptive controller.

Note also that in the continuous-time adaptive case, the closed-loop performance is significantly worse in the presence of noise corrupting the temperature measurement than in the noise free case, whereas in the sampled-data case the difference in performance with and without noise is not that significant, see Figure 2. One reason for this could be that the continuous-time noise

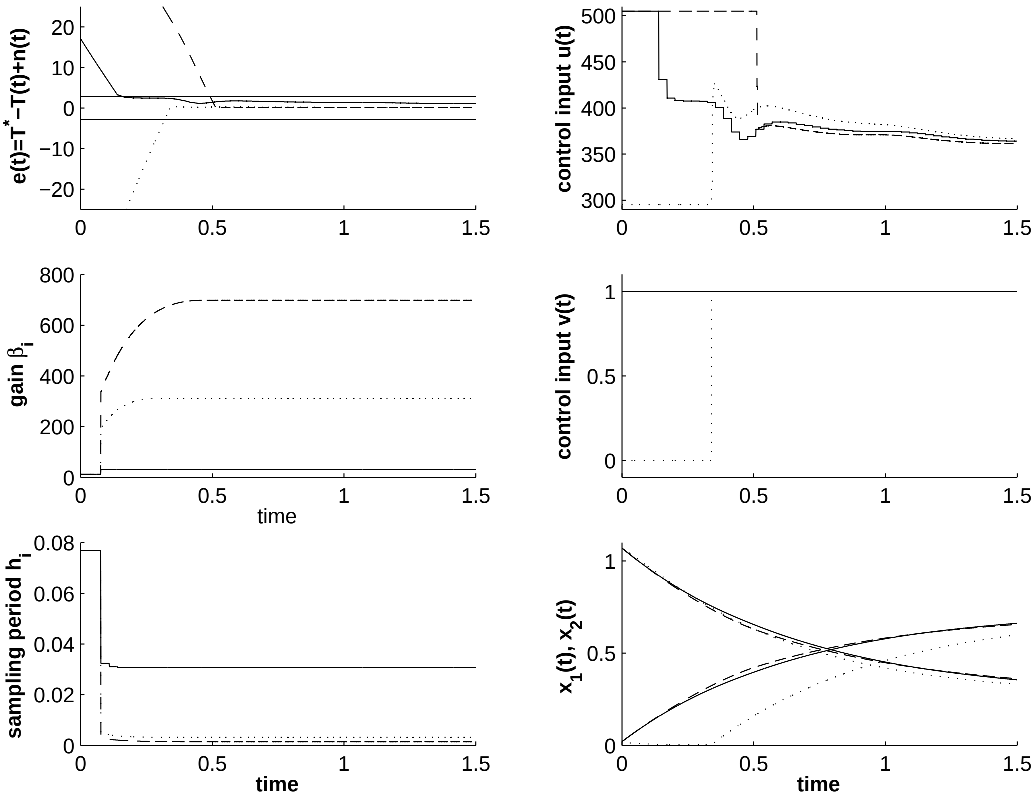

Fig. 2. Closed-loop setpoint control of the adaptive λ-tracker (7) with control parameters (12) applied to the single reaction (10) with parameters (11), in the presence of measurement noise (14), and initial temperature $T^0 = 270$ (dashed), $T^0 = 320$ (solid), $T^0 = 390$ (dotted).

might fluctuate much more wildly than its sampled-data counterpart.

4 Proof of Theorem 1

It is worth emphasising that there are a number of essential differences between the proof of the sampled-data result of Theorem 1 and the analogous continuous-time result in Theorem 10 in [5]. Indeed, whilst in the continuous-time case it is easy to see that the solutions exists for all time, here we have to rule out the possibility that the sampling periods have a finite-sum, i.e. $\sum_{i=0}^{\infty} h_i < \infty$, so that the solution only exists on a finite interval. A more awkward problem arises because of the availability of certain signals only at sampling instants. In the continuous-time proof we appeal repeatedly to the fact that certain temperature values are 'repelling' by using a Lypunov-function argument. In the sampled-data proof we can no longer appeal to this idea. Instead we need to work with 'repelling' intervals.

Let $n(\cdot) : \mathbb{R}_{\geq 0} \to \mathbb{R}$ denote a continuous noise satisfying (8). We may choose $\eta > 0$ such that

$$2\,\|n\|_\infty + \eta \;<\; \overline{T} - T^* \,. \tag{16}$$

We proceed in several steps.

STEP 1: We show positive invariance of $\Omega(\gamma, x^{\text{in}}) \times \mathbb{R}_{>0}$ under (1).
By (A1), we may consider a solution $(x, T) : [0, \omega) \to \mathbb{R}^n_{\geq 0} \times \mathbb{R}_{>0}$ of (1) for some $\omega \in (0, \infty]$ and with initial data $(x(0), T(0)) \in \Omega(\gamma, x^{\text{in}}) \times \mathbb{R}_{>0}$. Then integrating

$$\tfrac{\mathrm{d}}{\mathrm{d}\tau}\, \gamma^T x(\tau) = \gamma^T C r(x(\tau), T(\tau)) + d\gamma^T \left[\begin{pmatrix} v(\tau) \\ x_2^{\text{in}} \end{pmatrix} - x(\tau) \right]$$

over $[0, t)$, for arbitrary $t \in [0, \omega)$, yields, by invoking (A2) and $\gamma^T x(0) < \gamma^T x^{\text{in}}$,

$$\begin{aligned} \gamma^T x(t) \;&\leq\; e^{-dt} \gamma^T x(0) + d \int_0^t e^{-d(t-\tau)} \mathrm{d}\tau \; \gamma^T x^{\text{in}} \\ &\leq\; \gamma^T x^{\text{in}} - e^{-dt} [\gamma^T x^{\text{in}} - \gamma^T x(0)] \;\leq\; \gamma^T x^{\text{in}}. \end{aligned}$$

This proves the claim.

STEP 2: We show that the initial value problem (1), (3)–(6), $(x(0), T(0)) = (x^0, T^0)$, has a unique solution

$$(x, T) : [0, \infty) \to \Omega(\gamma, x^{\text{in}}) \times \mathbb{R}_{>0} \,. \tag{17}$$

(2A): By (A1) and since the right hand side of (1) is locally Lipschitz in its arguments, the theory of ordinary differential equations ensures, for any initial data $(x^0, T^0) \in \Omega(\gamma, x^{\text{in}}) \times \mathbb{R}_{>0}$, the existence of a unique solution $(x, T) : [0, \delta_1) \to \Omega(\gamma, x^{\text{in}}) \times \mathbb{R}_{>0}$ for some maximal $\delta_1 \in (0, t_1]$. Since x is bounded by Step 1, the affine linear boundedness of the right hand side of (1) in T ensures that the T-dynamics cannot produce a finite escape time. Therefore, $\delta_1 = t_1$.
(2B): Consider next the initial value problem (1), $(\tilde{x}(0), \tilde{T}(0)) = (x^0, T^0)$ with $u(\cdot) \equiv u_0$, $v(\cdot) \equiv v_0$ on $\mathbb{R}_{\geq 0}$. With the same reasoning as in Step 2A, it follows that there exists a unique solution $(\tilde{x}, \tilde{T}) : [0, \infty) \to \Omega(\gamma, x^{\text{in}}) \times \mathbb{R}_{>0}$ and, in particular, $(x(t), T(t)) = (\tilde{x}(t), \tilde{T}(t))$ for all $t \in [0, t_1)$. Therefore $(x_1, T_1) := \lim_{t \to t_1} (x(t), T(t))$ is well defined.
(2C): Analogous reasoning as in Step 2A and 2B for the initial value problem (1), (3)–(6), $(x(t_1), T(t_2)) = (x_1, T_1)$ extends the solution (x, T) uniquely on $[0, t_2)$. Proceeding inductively in this way proves that the there is a unique and maximal solution

$$(x, T) : [0, \omega) \to \Omega(\gamma, x^{\text{in}}) \times \mathbb{R}_{>0} \qquad \text{for} \quad \omega := \sum_{i=0}^{\infty} h_i \in (0, \infty] \tag{18}$$

of the initial value problem (1), (3)–(6), $(x(0), T(0)) = (x^0, T^0)$.
(2D): We show that $\omega = \infty$ in (18).
Seeking a contradiction suppose that $\omega < \infty$. Finiteness of ω in turn implies that $\sum_{i=0}^{\infty} h_i < \infty$ which in turn means that $\lim_{i\to\infty} h_i = 0$. Since $\dot{T}$ is affine linearly bounded in T, see (1), it follows that T is bounded on $[0, \omega)$. Therefore, (T_i) is bounded and thus there exists $M > 0$ so that $g_\lambda(e_i) \le M$ for all $i \in \mathbb{N}$. Using (5), this implies that

$$\beta_{i+1} \;\le\; \beta_0 + M \sum_{j=0}^{i} h_j \qquad \text{for all } i \in \mathbb{N}\,,$$

which yields boundedness of (β_i), and thus $(h_i) = (f(\beta_i))$ is uniformly bounded away from 0, which contradicts the assumption. This proves Step 2D and completes the proof of Step 2.

STEP 3: We show that there exists $M > 0$ such that

$$e^{-qh_i} T_i \;\le\; T(t) \;\le\; e^{Mh_i} T_i + \frac{e^{Mh_i} - 1}{M} \overline{u} \quad \forall\; i \;\in\; \mathbb{N} \;\forall\; t \;\in\; [t_i, t_{i+1})\,.$$

Since x_1 lies in a bounded set, continuity of $\hat{r}$ ensures the existence of $M > 0$ such that

$$-qT(t) + \underline{u} \;\le\; \dot{T}(t) \;\stackrel{(A4)}{\le}\; \|b\| \hat{r}(x_1)\, T(t) - q\, T(t) + \overline{u} \;\le\; M\, T(t) + \overline{u}\,,$$

and integration gives, for all $i \in \mathbb{N}$ and all $t \in [t_i, t_{i+1})$,

$$e^{-q(t-t_i)} T(t_i) + \int_{t_i}^{t} e^{-q(t-s)} \underline{u}\, \mathrm{d}s \le T(t) \le e^{M(t-t_i)} T(t_i) + \int_{t_i}^{t} e^{M(t-s)} \overline{u}\, \mathrm{d}s\,,$$

and so the claim of Step 3 follows.

STEP 4: We show that boundedness of (β_i) yields Assertions (i)–(iv).
If (β_i) is bounded, then by monotonicity we have (ii); and furthermore $(h_i) = (f(\beta_i))$ is uniformly bounded away from 0, whence (iii). Now (5) ensures that

$$h_\infty \sum_{i=0}^{\infty} g_\lambda(e_i) \;\le\; \sum_{i=0}^{\infty} h_i\, g_\lambda(e_i) \;=\; \beta_\infty - \beta_0 \in \mathbb{R}.$$

Therefore $\lim_{i\to\infty} g_\lambda(e_i) = 0$, and (6a), (6b) give $\lim_{i\to\infty} \operatorname{dist}(|e_i|, [0, \lambda]) = 0$, which proves (iv). We then have that (T_i) is bounded, and therefore boundedness of $T(\cdot)$ on $[0, \infty)$ is a consequence of Step 3, whence (i).

STEP 5: If (β_i) is unbounded, then we may choose $i_0 \in \mathbb{N}$ so that

$$\beta_i \ge \max\left\{ \frac{2(u^* - \underline{u})}{\lambda - 2\|n\|_\infty}, \frac{2(\overline{u} - u^*)}{\lambda - 2\|n\|_\infty}, \frac{u^* - \underline{u}}{\eta} \right\} \qquad \forall\; i \ge i_0 \qquad (19)$$

holds. Note that if (19) does not hold for any $i_0 \in \mathbb{N}$, then, by invoking that (β_i) is a non-decreasing sequence, it follows that (β_i) is bounded, and (i)–(iv) are a consequence of Step 4.

STEP 6: We show that if (β_i) is unbounded, then there exists $i_1 \geq i_0$ such that $T_{i_1} \in (0, \overline{T}]$.
Suppose (β_i) is unbounded and let i_0 be such that (19) holds.

(6A): We show

$$\Big[i \geq i_0 \ \wedge \ T_i > \overline{T}\Big] \quad \Longrightarrow \quad \Big[\forall\, t \in [t_i, t_{i+1}) : \ u(t) = \underline{u} \ \wedge v(t) = 0\Big]. \tag{20}$$

Indeed, if $T_i > \overline{T}$ and $i \geq i_0$, then

$$\beta_i e_i \ + \ u^* \ \overset{(2)}{\leq} \ \beta_i [T^* \ - \ \overline{T} \ + \ \|n\|_\infty] \ + \ u^* \ \overset{(16)}{\leq} \ -\beta_i \eta \ + \ u^* \ \overset{(19)}{\leq} \ \underline{u}$$

which, applied to (6), yields (20).
(6B): We show that

$$\Big[\forall\, t \in [t_i, t_{i+1}) : \ v(t) = 0\Big] \\ \Longrightarrow \quad \Big[\forall\, t \in [t_i, t_{i+1}) : \ \|x_1(t)\| \ \leq \ e^{-d(t-t_i)} \|x_1(t_i)\|\Big]. \tag{21}$$

Since all entries of C^1 are non-positive, (1) gives, for all $\tau \in [t_i, t_{i+1})$,

$$\tfrac{\mathrm{d}}{\mathrm{d}\tau} \|x_1(\tau)\|^2 = 2\, x_1(\tau)^T \Big[C^1\, r\Big(x(\tau), T(\tau)\Big) - d\, x_1(\tau)\Big] \leq -2\, d\, \|x_1(\tau)\|^2, \tag{22}$$

and the claim follows by integration over $[t_i, t)$.
(6C): Finally, seeking a contradiction to the claim of Step 6, assume that $T_i > \overline{T}$ for all $i \geq i_0$.

By (A3), we may choose $\varepsilon \in (0, q)$ sufficiently small so that

$$-\,[q - \varepsilon]\, \overline{T} + \underline{u} \ < \ -\rho\,. \tag{23}$$

Note that (20) together with (21) yields

$$\|x_1(t)\| \ \leq \ e^{-d(t-t_{i_0})} \|x_1(t_{i_0})\| \qquad \text{for all} \ \ t \in [t_{i_0}, \infty)\,,$$

and hence, by (A4), there exists $t^* \geq t_{i_0}$ so that

$$\hat{r}(x_1(t)) \ \leq \ \varepsilon / \|b\| \qquad \text{for all} \quad t \geq t^*\,. \tag{24}$$

Therefore,

$$\begin{aligned}\dot T(t) &\overset{(20)}{=} b^T r\Big(x(t),T(t)\Big) - qT(t) + \underline{u} \\ &\qquad\qquad \le \|b\|\,\hat r(x_1(t))\,T(t) - qT(t) + \underline{u} \overset{(24)}{\le} -[q-\varepsilon]\,T(t) + \underline{u}\,,\end{aligned}$$

and hence it follows for $t \ge t^*$ that whilst $T(t) > \overline{T}$, we have

$$\dot T(t) \le -[q-\varepsilon]\,\overline{T} + \underline{u} \overset{(23)}{<} -\rho\,.$$

This contradicts the assumption and completes the proof of Step 6.

Step 7: We show that

$$\exists i_1 \ge i_0 \;:\Big[T_{i_1} \in (0,\overline{T}] \quad\Longrightarrow\quad \Big[\forall\, t \in [t_{i_1},\infty):\; T(t) \in (0,\overline{T}]\Big]\Big]. \qquad (25)$$

(7A): Let $i \ge i_0$ and $T_i \in [\overline{T} - \|n\|_\infty, \overline{T}]$. We show that if $t \in [t_i, t_{i+1})$ and $T(t) \in [\overline{T} - \|n\|_\infty, \overline{T}]$, then $\dot T(t) < -\rho$. Indeed, since $\overline{T} - \|n\|_\infty \le T_i$ we have

$$\begin{aligned}\beta_i e_i + u^* &\overset{(2)}{\le} \beta_i[T^* - T_i + \|n\|_\infty] + u^* \\ &\qquad \le \beta_i[T^* - \overline{T} + 2\|n\|_\infty] + u^* \overset{(16)}{\le} -\beta_i\eta + u^* \overset{(19)}{\le} \underline{u}\,,\end{aligned}$$

so that

$$\dot T(t) \;=\; b^T r\Big(x(t),T(t)\Big) - qT(t) + \underline{u} \qquad \text{for all}\;\; t \in [t_i, t_{i+1})\,. \qquad (26)$$

If $\overline{T} - \|n\|_\infty \le T(t)$, then (8) yields $T^* + 2\|n\|_\infty < \overline{T} \le T(t) + \|n\|_\infty$, and so $\underline{T} \le T^* + \|n\|_\infty < T(t) \le \overline{T}$, and we may apply (A3) to (26) to conclude that $\dot T(t) < -\rho$.

(7B): Finally, to show Step 7, assume that there exists $i \ge i_0$ such that $T_i \in (0,\overline{T}]$ and consider two cases:

(i) If $T_i \in [\overline{T} - \|n\|_\infty, \overline{T}]$, then Step 7A ensures that $T(t)$ moves to the left if $t \in [t_i, t_{i+1})$ and $T(t) \in [\overline{T} - \|n\|_\infty, \overline{T}]$. Therefore, $T(t) \le \overline{T}$ over the period $[t_i, t_{i+1})$.

(ii) Suppose that $T_i \in (0, \overline{T} - \|n\|_\infty]$. By Step 3, one may choose $h > 0$ such that

$$\Big[\exists\, t^* \ge 0 :\; T(t^*) \in (0, \overline{T} - \|n\|_\infty]\Big] \;\Longrightarrow\; \Big[\forall\, t \in [t^*, t^*+h) :\; |T(t) - T(t^*)| < \|n\|_\infty\Big].$$

Now one may choose $i_1 \in \mathbb{N}$ such that $h_i < h$ for all $i \ge i_1$. It then follows that

$$\Big[\forall\, i \ge i_1 :\; T_i \in (0, \overline{T} - \|n\|_\infty]\Big] \;\Longrightarrow\; \Big[T_{i+1} < \overline{T}\Big].$$

This completes the proof of Step 7.

Step 8: To show boundedness of (β_i), we prove that unboundedness of (β_i) yields

$$\exists\ i_3 \geq i_1\ \forall\ i \geq i_3\ :\ \beta_i = \beta_{i_3} \tag{27}$$

whence (β_i) is bounded.
Suppose (β_i) is unbounded. Then Step 5,6, and 7 guarantee that

$$\forall\ t \in [t_{i_1}, \infty) :\ T(t) \in (0, \overline{T}]\,. \tag{28}$$

By Step 3 there exists $h > 0$ such that

$$\forall\ h_i < h : \quad |T_{i+1} - T_i| < \lambda/2\,. \tag{29}$$

If $\lim_{i\to\infty} h_i \geq h$, then (β_i) is bounded, and (i)–(iv) are a consequence of Step 4. Hence we may suppose that there exists $i_2 \geq i_1$ such that $h_{i_2} < h$, which in turn means that $h_i < h$, for all $i \geq i_2$.
(8a): We show that

$$\left[\exists\ i \geq i_2 :\ T_i < T^* - \tfrac{\lambda}{2}\right] \quad \Longrightarrow \quad \left[\forall\ t \in [t_i, t_{i+1}) :\ \dot{T}(t) > \rho\right]$$

and

$$\left[\exists\ i \geq i_2 :\ T^* + \tfrac{\lambda}{2} < T_i\right] \quad \Longrightarrow \quad \left[\forall\ t \in [t_i, t_{i+1}) :\ \dot{T}(t) < -\rho\right].$$

Let $T_i < T^* - \lambda/2$ and $i \geq i_2$. Then, by (19), we have $\beta_i e_i + u^* \geq \overline{u}$, and so $u(t) = \overline{u}$ for all $t \in [t_i, t_{i+1})$, which gives, by invoking (28) and (A3),

$$\dot{T}(t) \ = \ b^T\, r\Big(x(t), T(t)\Big) - qT(t) + \overline{u} \ \geq \ \rho \qquad \forall\ t \in [t_i, t_{i+1})\,.$$

The second implication follows similarly.
(8b): Finally, if $i \geq i_2$ and $T_i \notin [T^* - \frac{\lambda}{2}, T^* + \frac{\lambda}{2}]$, then by Step 8a together with (29) it follows that there exists $i_3 \geq i_2$ such that $T_{i_3} \in (T^* - \frac{\lambda}{2}, T^* + \frac{\lambda}{2})$. Invoking (29) again yields $T_{i_3+1} \in (T^* - \lambda, T^* + \lambda)$, and therefore we have, for all $i \geq i_3$, that $T_i \in (T^* - \lambda, T^* + \lambda)$, whence (27). This completes the proof of Step 8 and also the proof of the theorem.

Acknowledgment: We are indebted to Thomas Müller (Ilmenau) for carefully performing the simulations.

References

[1] Bastin, G. and D. Dochain (1990). *On-Line Estimation and Adaptive Control of Bioreactors*. Elsevier Science Publishers B.V. Amsterdam and others.

[2] Gavalas, G.R. (1968). *Nonlinear Differential Equations of Chemically Reacting Systems*. Springer-Verlag. Berlin - Heidelberg - New York.

[3] Ilchmann, A. and M.-F. Weirig (1999). Modelling of general biotechnological processes. *Mathematical and Computer Modelling of Dynamical Systems* **5**, 152–178.

[4] Ilchmann, A. and S. Trenn (2004). Input constrained control with applications to chemical reactor models. *to appear in: Systems & Control Letters*.

[5] Ilchmann, A., M. Thuto and S. Townley (2004). Input constrained adaptive tracking with applications to exothermic chemical models. *SIAM J. of Control and Optim.* **43**, 154–173.

[6] Jadot, F., G. Bastin and F. Viel (1999). Robust global stabilization of stirred tank reactors by saturated output feedback. *European Journal of Control* **5**, 361–371.

[7] Sparrow, C. (1982). *The Lorenz Equations: Bifurcations, Chaos, and Strange Attractors*. Springer-Verlag. New York.

[8] Viel, F., F. Jadot and G. Bastin (1997). Global stabilization of exothermic chemical reactors under input constraints. *Automatica* **33**, 1437–1448.